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CIRCULAR PLATE UNDER A UNIFORMLY DISTRIBUTED IMPULSE*

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Abstract—Experiments are described in which simply supported circular plates are subjected to uniformly distributed impulses and the permanent deformations obtained are compared with those predicted by the bending theory of rigid-plastic plates. As opposed to the behavior of beams it is found that agreement between theory and experiment deteriorates as the impulse increases beyond a certain value especially when the predicted permanent central deflection is greater than about one-quarter of the plate radius. This illustrates the appreciable strengthening of plates due to membrane forces not taken into account in the theory.

INTRODUCTION

IN THIS paper the description and results of experiments are presented in which simply supported circular plates are subjected to uniformly distributed impulses. The results in the form of final permanent deformations are compared with those predicted by the bending theory of rigid-plastic plates given by Wang [1].

Many problems concerning the dynamic response of beams and plates have simple solutions when the material is assumed to behave in a rigid-perfectly plastic manner [2]. Consequently, they have an appeal for engineering applications. It remains, however, to evaluate many of the solutions by means of experiments. Some progress has been made in this direction, especially for beams [3, 4, 5, 6].

The results of beam experiments [3, 5] show that correlation is quite good whenever the ratio R of the kinetic energy input to the elastic strain energy capacity is greater than 2 to 3, and it improves as R increases. This means that the assumption of neglecting the elastic deformation or strain energy is reasonable if the plastic deformation or work is large enough. The results of the plate experiments given later show that the rigid-plastic theory is again a reasonable first-order theory whenever the energy ratio R is greater than about 4. However, unlike the beams, correlation does not improve as R increases beyond a certain value and this is due to the deformations becoming large enough to bring membrane forces into play. The plates show considerable increase in strength when the central deflection exceeds about one-fifth of the plate radius (or about one-quarter using the predicted values).

THEORY

The solution of Wang [1] involves a material obeying the Tresca yield condition and the associated flow rule. A physical picture of the mechanism of deformation is as follows. Immediately after the impulse has been applied a hinge circle is assumed created at the support of the plate of radius a whereupon it travels toward the plate center. Figure 1

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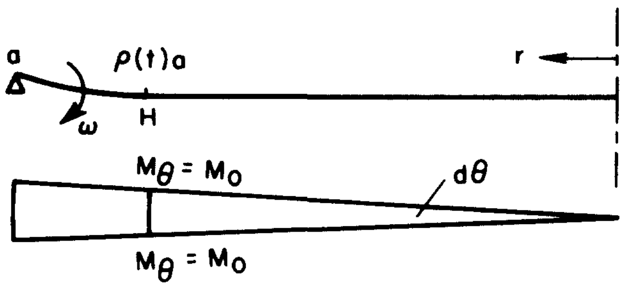

FIG. 1. Plate sector during phase 1.

shows the hinge circle H with radius $r = \rho(t)a$. The region inside the hinge circle moves at constant velocity $V = I/m$, where I and m are the impulse and mass per unit area of plate, until $\rho = 0$. During this time elemental segments of plate outside the hinge circle rotate as rigid bodies about the support with an angular velocity $\omega = V/a(1-\rho)$. Also the circumferential component of the bending moment M_θ equals the fully plastic moment M_0 while the radial component M_r increases from zero at the support to M_0 at the hinge circle. Equations expressing the conservation of linear momentum of the elemental segments inside the hinge circle and angular momentum about the support of elemental segments outside the hinge circle, along with the condition of a continuous velocity field, determine the motion during this phase. After the hinge circle reaches the center of the plate the whole elemental sector is assumed to rotate about the support as a rigid body. The angular momentum equation governs the motion. Using the above two phases of motion the permanent deflection of the plate is found by Wang to be

$$w = \frac{I^2a^2}{24mM_0}\left(1-\frac{r}{a}\right)\left[3+2\frac{r}{a}+\left(\frac{r}{a}\right)^2\right] \tag{1}$$

from which the central deflection δ is obtained by setting $r = 0$. Hence

$$\delta = I^2a^2/8mM_0. \tag{2}$$

Formulas (1) and (2) are used for comparison with experimental results.

An interesting observation that can be made from the analysis concerns the partitioning of the plastic work done between the above two phases of motion. Briefly, the plastic work may be expressed in the form

$$W_p = 2\pi M_0\int_0^1 (\varkappa_r+\varkappa_\theta)\rho \, d\rho \tag{3}$$

where $\varkappa_r = -w_{\rho\rho}/a^2$ and $\varkappa_\theta = -w_\rho/\rho a^2$ are the radial and circumferential components of curvature with $\rho = r/a$. The subscript ρ denotes partial differentiation. Substituting these curvature expressions in (3) gives

$$W_p = -2\pi M_0\int_0^1 (\rho w_\rho)_\rho \, d\rho = -2\pi M_0 w_\rho(1). \tag{4}$$

At the end of the first phase [1]

$$w = I^2a^2(1-\rho)(1+\rho+\rho^2/2)/12mM_0 \tag{5}$$

and at the end of the second phase

$$w = I^2a^2(1-\rho)(3+2\rho+\rho^2)/24mM_0. \tag{6}$$

From (5), $w_\rho(1) = -5I^2a^2/24mM_0$ and from (6), $w_\rho(1) = -I^2a^2/4mM_0$. When these results are substituted in (4) the plastic work done of the first and second phases are found to be $5\pi I^2a^2/12m$ and $\pi I^2a^2/2m$ (this latter work done equals the kinetic energy input) so the partitioning of the work done is $\frac{5}{6}$ and $\frac{1}{6}$. In the analogous simply-supported beam problem [2] the corresponding partitioning is $\frac{2}{3}$ and $\frac{1}{3}$.

Before turning to the experiments the expression will be derived for the ratio R of the kinetic energy input to the elastic strain energy capacity of the plate. Let the maximum elastic bending moment per unit length be M_e. Then $M_e = \sigma d^2/6$ where σ is the yield stress and d is the plate thickness. If this moment is applied uniformly around the circumference of the plate a state of pure bending exists. This is the state of maximum bending strain energy which, per unit area, is $M_e^2/(1+\nu)D$. $D = Ed^3/12(1-\nu^2)$ is the flexural rigidity, ν is Poisson's ratio, and E is Young's modulus. The kinetic energy delivered per unit area is $I^2/2m$ where I and m are the applied impulse and mass per unit area of plate. The energy ratio is therefore $R = 3I^2E/2\rho\sigma^2d^2(1-\nu)$ where ρ is the mass density ($m = \rho d$).

DESCRIPTION OF EXPERIMENTS

The experiments were performed with plates of 6061-T6 aluminum and 1018 cold-rolled steel, all nominally $\frac{1}{4}$-inch thick and $8\frac{1}{2}$ inches in diameter. They were simply supported on a heavy steel annulus at a diameter of 8 inches. Figures 2 and 3 show the experimental arrangement. The impulse was generated by sheet explosive* rolled to a uniform thickness and cut out to form a disk of 8 inches in diameter. This was placed over a similar disk of solid neoprene attenuator nominally $\frac{1}{8}$-inch thick which in turn was layed centrally

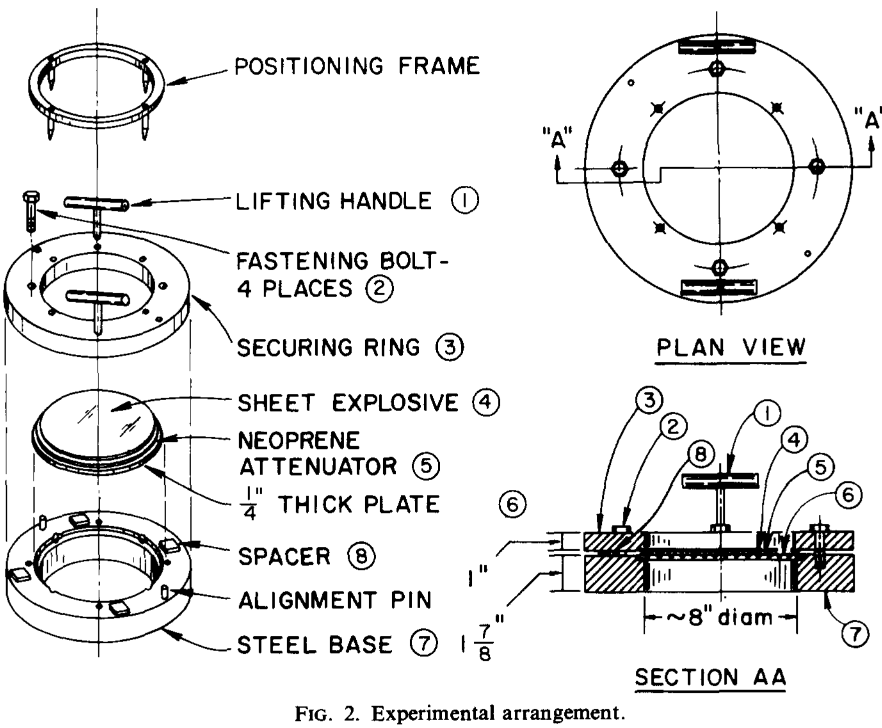

FIG. 2. Experimental arrangement.

* DuPont EL-506D.

over the plate. The neoprene was used to reduce the high peak pressure in the shock wave entering the plate to eliminate plastic waves, possible change of material properties, and spalling. Five grain mild detonating fuse was used to detonate the explosive. The detonation velocity (0·27 in/μsec) is supersonic relative to the plate wave velocities (0·21 in/μsec maximum) and the initiation point is at the center of the plate so it is assumed, by analogy with the beam results [7], that simulation of an ideal impulse simultaneously applied over the whole plate is satisfactory. As can be seen in Figs. 2 and 3 a steel annulus is placed over the supporting annulus to control the plate as it rebounds. Sufficient clearance was provided between the two annuli by means of spacers to prevent the edge of the plate striking the upper annulus as it deforms plastically.

For the explosive-attenuator-target configuration described above, the impulse imparted was obtained by firing free plates in front of a double-flash X-ray unit. The rigid body displacement in the predetermined time between the X-ray pictures gives the plate velocity. It was found that the velocity imparted was proportional to the thickness of the explosive over a range from 15 to 60 mils, the range of interest in the actual plate experiments. This procedure thus provided a simple linear calibration curve of impulse versus explosive thickness. The constant slope of this curve is expressible as impulse per unit volume of explosive with units dyne-sec/cm^2/mil or dyne-sec/cm^3 and is given the symbol I_0. (Values of I_0 for aluminum and steel are listed in Table 1.)

TABLE 1. PROPERTIES

Material	Modulus (lb/in^2) E	Yield stress (lb/in^2) σ	Mass density (lb sec^2/in^4) ρ	Plate depth (inch) d	Plate radius (inches) a	Impulse constant (dyne-sec/cm^3) I_0
Al 6061-T6	10×10^6	42,000	0·000253	0·251	4	$2·5 \times 10^5$
C.R. steel 1018	30×10^6	79,000	0·000732	0·241	4	$2·7 \times 10^5$

To determine the yield stress an average value was taken of static tensile tests with specimens cut with and across the grain. Each stress–strain curve was replaced by two straight lines, the slope of the strain-hardening portion being obtained by curve fitting to about 3 to 4 per cent strain. The ordinate of their point of intersection was taken as the yield stress.

The materials in Table 1 were chosen partly because of the small strain-hardening moduli and because they are believed to be somewhat insensitive to strain rate.

In addition to the permanent central deflection of each plate the deflection of points along a radius were measured using a travelling microscope. In the following, these data are given and are compared with the predictions of the theoretical formulas (1) and (2).

EXPERIMENTAL RESULTS AND OBSERVATIONS

Table 1 contains the materials and properties and the impulse calibration constant I_0 mentioned earlier. Table 2 contains the results of the experiments.

From formula (2) and the values in Table 2, Figs. 4 and 5 are drawn showing the variation of the permanent central deflection with impulse. Figures 6 and 7 show the variation

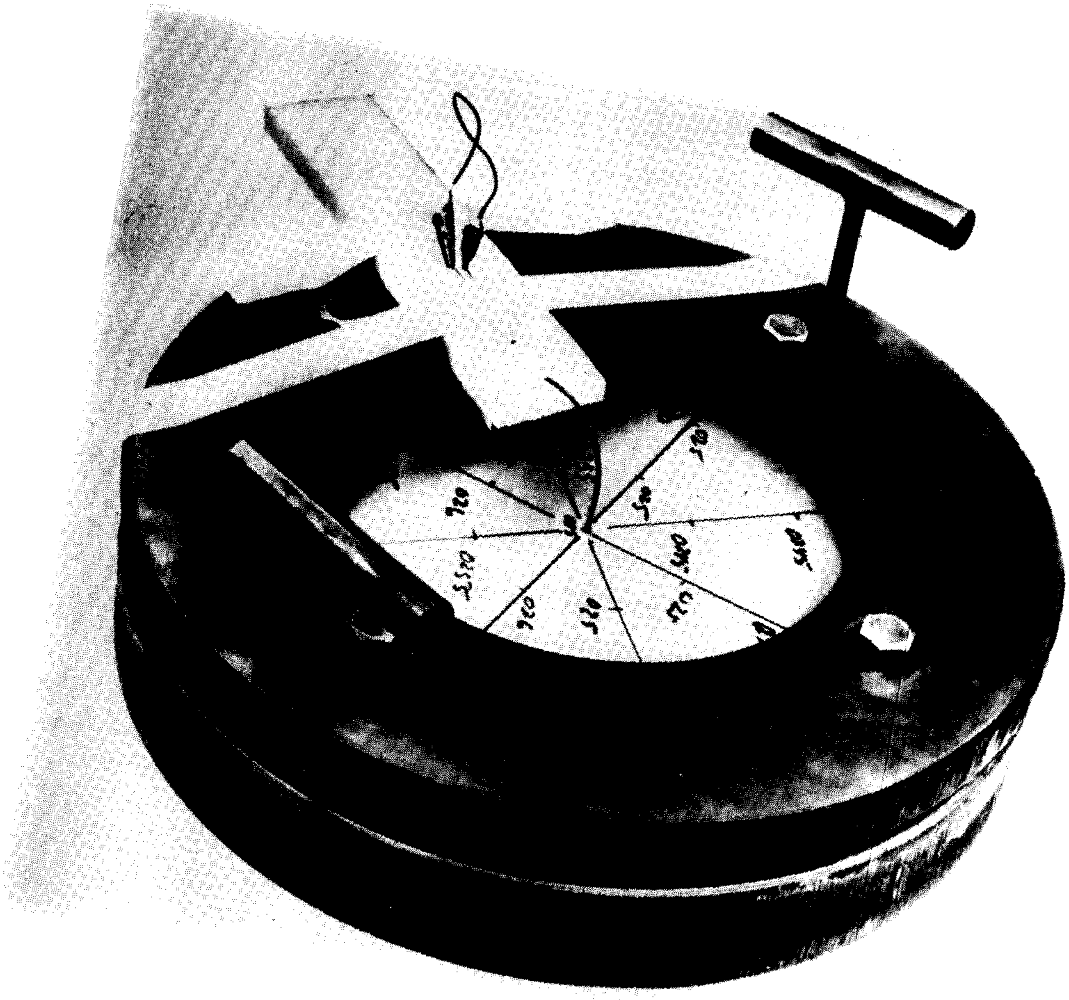

FIG. 3. Experimental arrangement.

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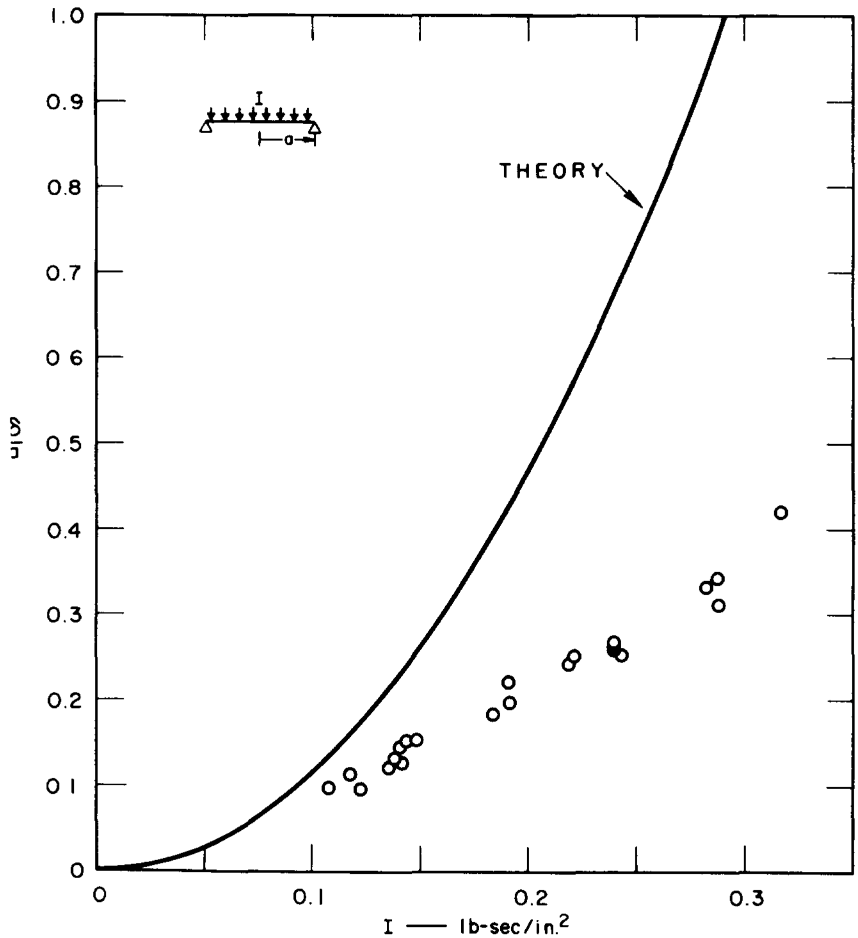

FIG. 4. Impulse versus central deflection—Al 6061-T6.

of the ratio of experimental to theoretical central deflection δ_{ex}/δ_{th} with impulse I and energy ratio R. Figures 8 and 9 show, in non-dimensional form, the theoretical shape of the plate according to formula (1) and a few of the experimentally obtained shapes.

To illustrate the difference between the plate and the corresponding simply-supported beam as regards the trend of the central deflections as the impulse is increased, Figs. 10 and 11 have been drawn. They are in non-dimensional form and Fig. 11 was obtained from [5]. A comparison is made subsequently.

Measurements of the plate thicknesses indicated thinning at the centers and thickening just inside the supporting circles. In the series of aluminum plates the extent of thinning increased gradually (with increasing impulse) to 8 per cent and that of thickening to 6 per cent. In the series of steel plates the corresponding values were 4 per cent and 4 per cent.

The main observation to be made is that within certain limits to be described the rigid-plastic theory does serve as a reasonable first order theory; experimental central deflections are about 60 per cent of the theoretical central deflection ($\delta_{ex}/\delta_{th} \approx 0{\cdot}6$) and, as can be seen in Figs. 8 and 9, the deformed plate is described fairly adequately. The lower limit of the useful range is determined by the energy ratio R. In the present series

TABLE 2. EXPERIMENTAL RESULTS

Material	Experiment No.	Impulse (lb sec/in^2) I	Energy ratio* R	δ_{ex}/a	δ_{th}/a	δ_{ex}/δ_{th}
Al 6061-T6	1	0·317	76·5	0·421	1·195	0·352
	2	0·289	63·7	0·312	0·994	0·314
	3	0·288	63·3	0·344	0·989	0·348
	4	0·283	61·2	0·333	0·956	0·348
	5	0·244	45·2	0·253	0·706	0·358
	6	0·240	44·1	0·268	0·688	0·389
	7	0·240	44·1	0·261	0·688	0·380
	8	0·240	43·8	0·264	0·684	0·387
	9	0·221	37·1	0·253	0·579	0·437
	10	0·219	36·7	0·243	0·573	0·425
	11	0·192	28·1	0·199	0·438	0·455
	12	0·191	27·7	0·222	0·433	0·514
	13	0·184	25·8	0·188	0·403	0·467
	14	0·149	16·9	0·155	0·264	0·588
	15	0·144	15·8	0·152	0·247	0·615
	16	0·142	15·3	0·127	0·239	0·533
	17	0·141	15·1	0·147	0·235	0·625
	18	0·139	14·6	0·134	0·228	0·588
	19	0·136	14·1	0·122	0·221	0·551
	20	0·123	11·6	0·098	0·181	0·541
	21	0·118	10·6	0·116	0·165	0·700
	22	0·108	8·9	0·099	0·139	0·715
C.R. Steel 1018	1	0·505	61·7	0·261	0·629	0·414
	2	0·501	60·8	0·254	0·620	0·410
	3	0·450	49·0	0·224	0·500	0·448
	4	0·436	46·1	0·215	0·471	0·456
	5	0·414	41·4	0·211	0·423	0·498
	6	0·359	31·3	0·193	0·319	0·603
	7	0·349	29·5	0·175	0·301	0·582
	8	0·344	28·7	0·167	0·292	0·571
	9	0·331	26·6	0·152	0·271	0·563
	10	0·314	23·9	0·135	0·243	0·553
	11	0·312	23·6	0·143	0·241	0·595
	12	0·272	17·9	0·114	0·183	0·623
	13	0·258	16·1	0·097	0·164	0·590
	14	0·215	11·2	0·077	0·114	0·674
	15	0·157	6·0	0·032	0·061	0·519
	16	0·156	5·9	0·031	0·060	0·507
	17	0·156	5·9	0·036	0·060	0·595
	18	0·153	5·6	0·045	0·058	0·786
	19	0·123	3·7	0·024	0·038	0·625
	20	0·121	3·6	0·025	0·036	0·676

* Value of Poisson's ratio is taken to be $\nu = 0{\cdot}3$.

of experiments, minimum values of R for aluminum and steel plates were about 9 and 4 respectively and the central deflection correlation was still satisfactory (Figs. 6 and 7). A value of $R = 4$ probably ensures a minor role for the elastic response of the aluminum as well as the steel plate. For the upper limit of the range of usefulness of the theory a suitable criterion is a maximum value for the ratio of the theoretical central deflection to the radius, found here to be $\delta_{th}/a \approx \frac{1}{4}$. This approximate ratio is the value where the experimental points plotted in Fig. 10 show a strong trend away from the theoretical line.

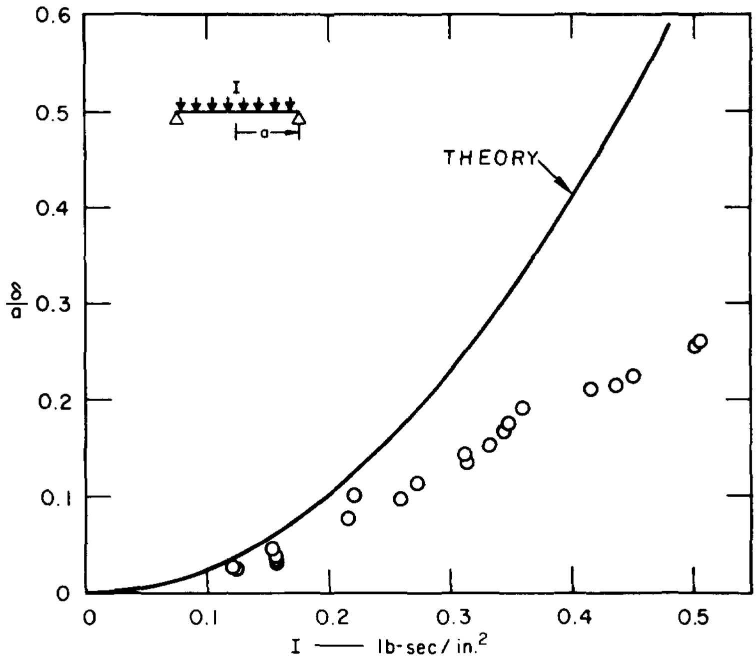

FIG. 5. Impulse versus central deflection—C.R. 1018 steel.

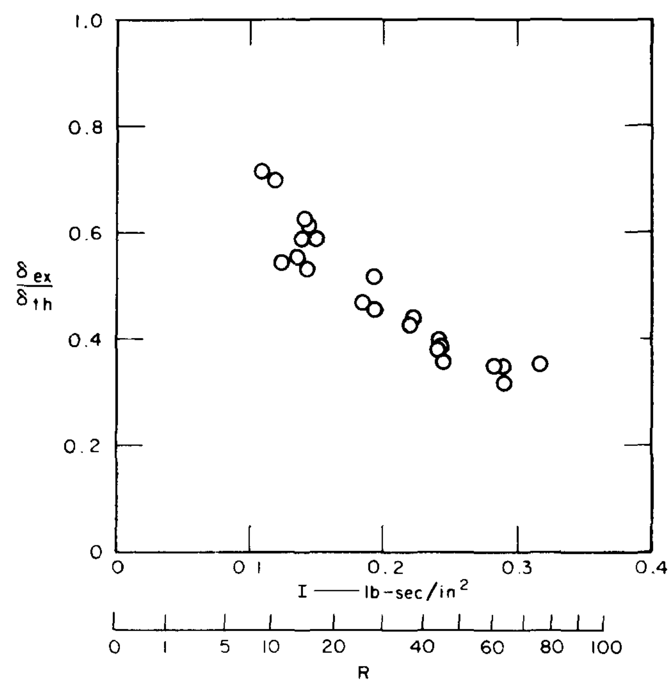

FIG. 6. Deflection ratio versus impulse and energy ratio—Al 6061-T6.

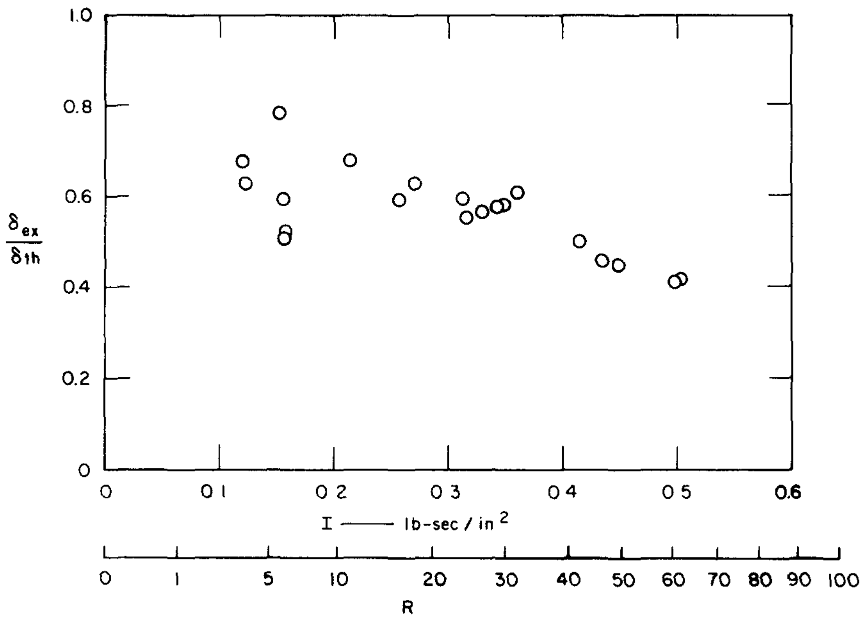

FIG. 7. Deflection ratio versus impulse and energy ratio—C.R. 1018 steel.

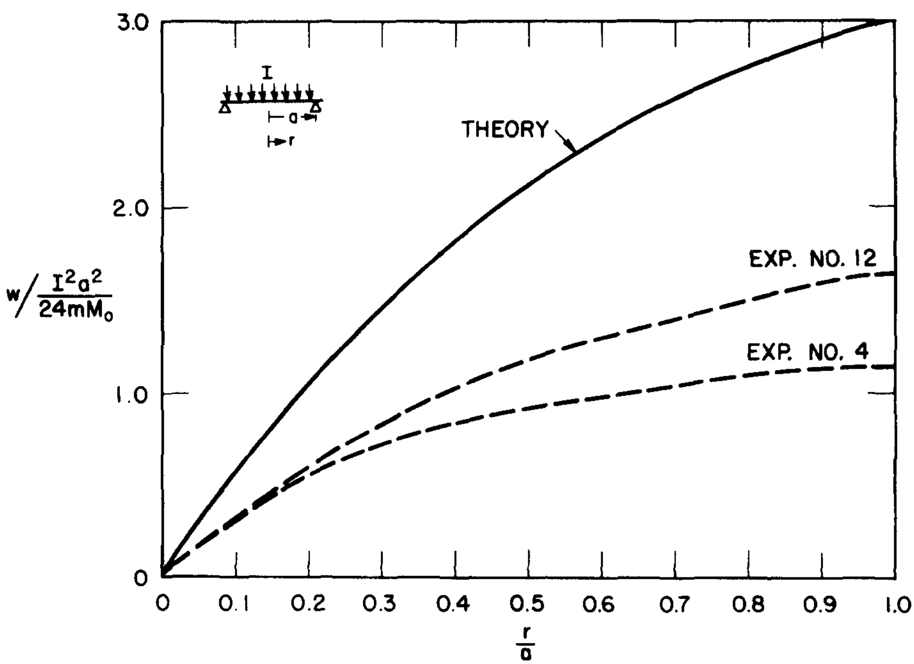

FIG. 8. Plate deflection curves—Al 6061-T6.

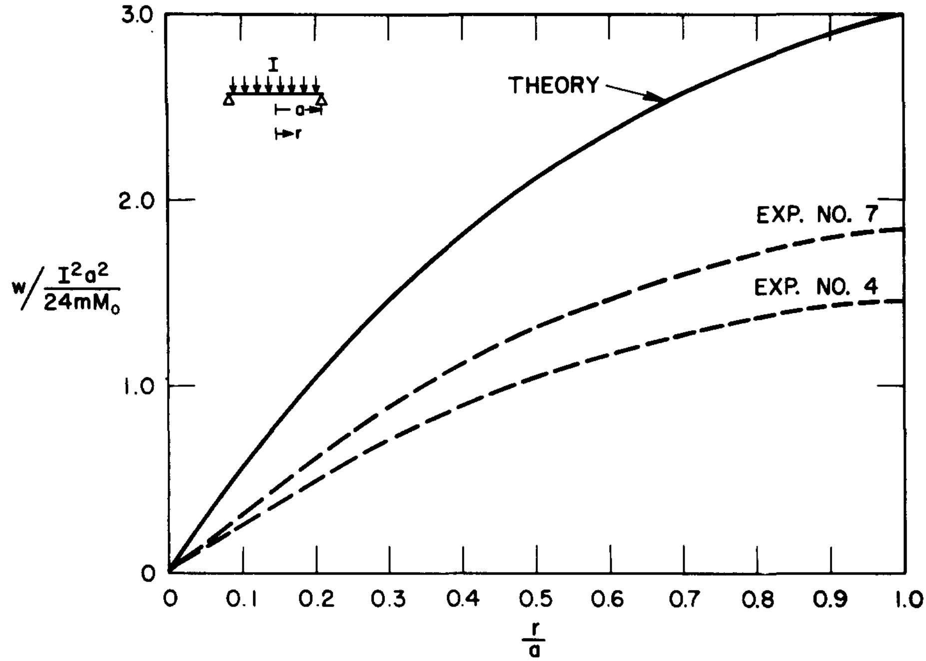

FIG. 9. Plate deflection curves—C.R. 1018 steel.

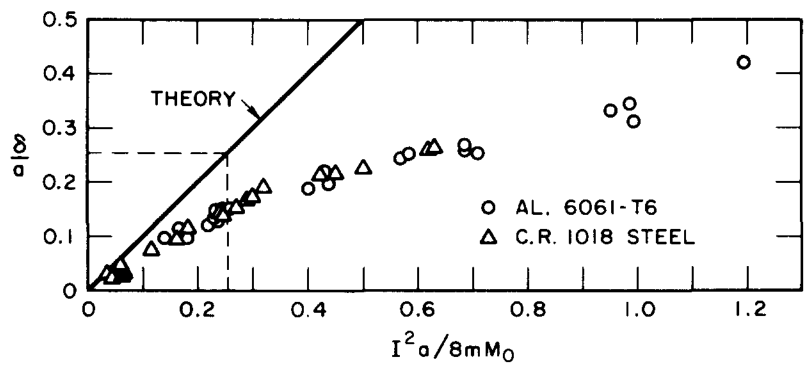

FIG. 10. Deflection–impulse relation for simply-supported plates.

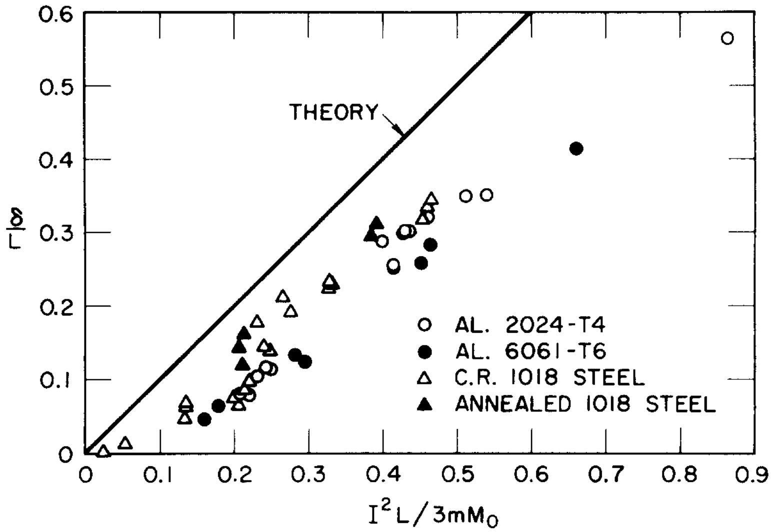

FIG. 11. Deflection–impulse relation for simply-supported beams.

No such break occurs in the trend of the experimental points for the corresponding simply-supported beam as can be seen in Fig. 11. Above this value correlation rapidly deteriorates due to the large deflections increasing the significance of the membrane forces.

Finally, from the above observations a full treatment of the problem taking account of the membrane action is certainly required.

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Résumé—Là sont décrites quelques expériences au cours desquelles des plaques circulaires, simplement supportées, sont soumises à des impulsions uniformément distribuées; les déformations permanentes obtenues sont alors comparées avec celles prévues par la théorie de la flexion des plaques plastiques-rigides. Contrairement au comportement des poutres, on a constaté que théorie et expérience deviennent de moins en moins conformes l'une à l'autre à mesure que les impulsions augmentent au-delà d'une certaine valeur, et ceci surtout lorsque la déformation centrale permanente prévue est supérieure au quart environ du rayon de la plaque. Ceci explique le renforcement notable des plaques, dû à des forces membraneuses qui ne sont pas prises en considération dans la théorie.

Zussammenfassung—Experimente werden beschrieben in denen einfach gestützte, kreisförmige Platten einheitlich verteilten Impulsen unterworfen wurden und die resultierenden permanenten Formveränderungen werden mit denen, welche durch die Biegungstheorie der starren plastischen Platten vorausgesagt wurde, verglichen.

Im Gegensatz zum Verhalten der Träger wurde gefunden, dass die Übereinstimmung von Theorie und Experiment loser wird wenn die Impulse einen bestimmten Wert übersteigen; insbesonders dann wenn die vorausbestimmte bleibende zentrale Durchbiegung grösser als ungefähr ein Viertel des Platten Halbmessers ist. Das ist die Erklärung für die merkliche Verstärkung der Platten welche Membrankräften zuzuschreiben ist welcher in der Theorie nicht Rechnung getragen wurde.

Абстракт—Описаны опыты в которых просто поддерживаемые круглые пластинки подвергаются равнрмерно распределенным нмпульсам и полученные постоянные деформации сопоставляются с деформациями предсказанными теорией изгибания твердых пластичных пластин. Установленно, что для дисков (в противоположность к брускам) согласованность теории с практикой уменьшается пропорционально с повышением импульса за предел определенной величины, особенно когда предсказанная постоянная цеитральная девиация превышает одну треть диаметра диска. Это демонстрирует значительное упрочнение дисков вызванное мембрановыми силами не принятыми в учет теорией.